

Non-minimal $R^\beta F^2$ -Coupled Electromagnetic Fields to Gravity and Static, Spherically Symmetric Solutions

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Abstract

We investigate non-minimal $R^\beta F^2$ -type couplings of electromagnetic fields to gravity. We derive the field equations by a first order variational principle using the method of Lagrange multipliers. Then we present various static, spherically symmetric solutions describing the exterior fields in the vicinity of electrically charged massive bodies.

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I. INTRODUCTION

Non-minimal couplings between electromagnetic fields and gravity can be induced in regions of space-time such as the neighborhood of black holes or neutron stars where high intensities of electromagnetic and/or gravitational fields are present. In such cases, gravitational fields behave as a non-linear medium in which the electromagnetic fields propagate. On the other hand, very intense electromagnetic fields may cause new gravitational effects.

Non-minimal couplings of the form RF^2 were investigated to obtain more information on the relationship between space-time curvature and electric charge conservation [1], [2]. They were also derived by Kaluza-Klein reduction of five-dimensional R^2 -Lagrangians to four dimensions [3–5]. Similar non-minimal coupling terms were obtained by calculating QED one-loop polarization on a curved background [6]. Recently, the behavior of the rotational velocities of test particles gravitating around galaxies was investigated in modified gravity for non-minimal matter couplings [7–9]. Furthermore, the non-minimal couplings of gravity and electromagnetism were extended to $I(R)F^2$ form to explain late-time cosmic acceleration [10, 11], primordial magnetic field during the reheating epoch [12] and rotation curves of galaxies[13].

Nevertheless, the non-minimal couplings have not been investigated in sufficient detail as regards to applications such as the contributions to the rotational curve of galaxies and Pioneer anomaly. To gain more insights on such configurations in non-minimal models, we proceed to investigate exact solutions of them for spherically symmetric systems with charge and mass. In order to find more general solutions which are compatible with observations from the solar system to cosmological scales, we propose $R^\beta F^2$ -type couplings between gravity and electromagnetism through arbitrary real power of the curvature scalar. We determine the field equations by a first order variational principle using the method of Lagrange multipliers. Then we present a wide range of exact static, spherically symmetric solutions that depend on various values of β . It is interesting to note that the solutions are asymptotically flat in general for values of $\beta \neq 0$ within the range $-\frac{1}{3} < \beta < 1$.

The non-minimal couplings in general give contributions both to the Maxwell and the Einstein field equations. Contributions to the Maxwell equations can be associated with the magnetization and the polarization of a specific medium while contributions to the Einstein equations may give important modifications to the space-time metric. Such effects,

if there are any, may be used to explain some unexpected observations of gravity such as dark matter, dark energy and Pioneer anomaly considering only the electromagnetic and gravitational fields.

II. FIELD EQUATIONS OF THE EXTENDED EINSTEIN-MAXWELL THEORY

We will obtain our field equations by a variational principle from an action

$$I[e^a, \omega^a_b, F] = \int_M L = \int_M \mathcal{L}^* 1 \quad (1)$$

where $\{e^a\}$ and $\{\omega^a_b\}$ are the fundamental gravitational field variables and F is the electromagnetic field 2-form. The space-time metric $g = \eta_{ab}e^a \otimes e^b$ with signature $(-+++)$ and we fix the orientation by setting $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. Torsion 2-forms T^a and curvature 2-forms R^a_b of space-time are found from the Cartan-Maurer structure equations

$$de^a + \omega^a_b \wedge e^b = T^a, \quad (2)$$

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b. \quad (3)$$

We consider the following Lagrangian density 4-form which involves $R^\beta F^2$ terms to all orders in β :

$$L = \frac{1}{2\kappa^2} R * 1 - \frac{1}{2} \sum_{n=0}^{\infty} ((a_0 R)^\beta)^n F \wedge *F \quad (4)$$

where κ^2 is Newton's universal gravitational coupling constant, a_0 is a coupling constant with dimension $[L]^2$ and β is a real number. When we take $a_0 = 0$, we get back to the minimal Einstein-Maxwell theory. If we take $a_0 \neq 0$ and assume that the condition $|(a_0 R)^\beta| < 1$ is satisfied, we can write $\sum_{n=0}^{\infty} ((a_0 R)^\beta)^n = \frac{1}{1 - (a_0 R)^\beta}$. The right hand side diverges as $\beta \rightarrow 0$ or $R \rightarrow \frac{1}{a_0}$. Therefore, the values $\beta = 0$ and $a_0 R = 1$ should be avoided. Thus we replace the Lagrangian (4) with

$$L = \frac{1}{2\kappa^2} R * 1 - \frac{1}{2(1 - (a_0 R)^\beta)} F \wedge *F. \quad (5)$$

which is meaningful even when the condition $|(a_0 R)^\beta| < 1$ is not satisfied.

The field equations are obtained by considering the independent variations of the action with respect to $\{e^a\}$, $\{\omega^a_b\}$ and $\{F\}$. The electromagnetic field components are read from the

expansion $F = \frac{1}{2}F_{ab}e^a \wedge e^b$. We will choose the unique metric-compatible, torsion-free Levi-Civita connection. We impose this choice of the connection through constrained variations by the method of Lagrange multipliers. We constrain the electromagnetic field 2-form F to be closed, that is; $dF = 0$, and this is imposed by the variation of a Lagrange multiplier 2-form μ . Therefore, we add to the above Lagrangian density the following constraint terms:

$$L_C = (de^a + \omega^a_b \wedge e^b) \wedge \lambda_a + dF \wedge \mu \quad (6)$$

where λ_a 's are Lagrange multiplier 2-forms whose variation imposes the zero-torsion constraint $T^a = 0$.

The infinitesimal variations of the total Lagrangian density $L + L_C$ (modulo a closed form) is found to be

$$\begin{aligned} \dot{L} + \dot{L}_C &= \frac{1}{2\kappa^2} \dot{e}^a \wedge R^{bc} \wedge *e_{abc} + \dot{e}^a \wedge \frac{1}{2(1 - (a_0 R)^\beta)} (\iota_a F \wedge *F - F \wedge \iota_a *F) + \dot{e}^a \wedge D\lambda_a \\ &+ \dot{e}^a \wedge \frac{\beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} (\iota_a R^b) (\iota_b F \wedge *F + F \wedge \iota_b *F) + \frac{1}{2} \dot{\omega}_{ab} \wedge (e^b \wedge \lambda^a - e^a \wedge \lambda^b) \\ &+ \dot{\omega}_{ab} \wedge \Sigma^{ab} - \dot{F} \wedge \frac{1}{1 - (a_0 R)^\beta} *F + \dot{\lambda}_a \wedge T^a - \dot{F} \wedge d\mu. \end{aligned} \quad (7)$$

where the angular momentum tensor Σ^{ab} :

$$\Sigma^{ab} = -\frac{1}{2} D \left[\frac{\beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} (F^{ab} *F + F^b \wedge \iota^a *F - F^a \wedge \iota^b *F - F \wedge \iota^{ab} *F) \right]. \quad (8)$$

We will use the abbreviations $e^a \wedge e^b \wedge \dots = e^{ab\dots}$, $\iota_a F = F_a$, $\iota_{ba} F = F_{ab}$, $\iota_a R^a_b = R_b$, $\iota_{ba} R^{ab} = R$. Lagrange multiplier 2-forms λ_a are solved from the connection variation equations

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = 2\Sigma_{ab} \quad (9)$$

by applying the interior product operators twice as

$$\lambda^a = 2\iota_b \Sigma^{ba} + \frac{1}{2} \iota_{bc} \Sigma^{cb} \wedge e^a. \quad (10)$$

When we substitute the λ_a 's above into the co-frame equation, we find the Einstein field equations for the non-minimal theory:

$$\begin{aligned} &\frac{1}{2\kappa^2} R^{bc} \wedge *e_{abc} + \frac{1}{2(1 - (a_0 R)^\beta)} (\iota_a F \wedge *F - F \wedge \iota_a *F) + D(2\iota_b \Sigma^{ba} + \frac{1}{2} e^a \wedge \iota_{bc} \Sigma^{cb}) \\ &+ \frac{\beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} (\iota_a R^b) (\iota_b F \wedge *F + F \wedge \iota_b *F) = 0. \end{aligned} \quad (11)$$

The Maxwell equations read

$$dF = 0 \quad , \quad d * \left(\frac{1}{(1 - (a_0 R)^\beta)} F \right) = 0. \quad (12)$$

We may represent the effects of non-minimal couplings of the electromagnetic fields to gravity through the definition of a constitutive tensor. Thus, Maxwell's equations for an electromagnetic field F in an arbitrary medium can be written as

$$dF = 0 \quad , \quad *d * G = J \quad (13)$$

where G is called the excitation 2-form and J is the source electric current density 1-form. The effects of gravitation and electromagnetism on matter are described by G and J . We can complete this system using electromagnetic constitutive relations relating G and J to F . Here we consider only the source-free interactions, that is $J = 0$. Then, we can write a simple linear constitutive relation

$$G = \mathcal{Z}(F) \quad (14)$$

where \mathcal{Z} is a type-(2,2)-constitutive tensor. For the above theory, we have

$$G = \frac{1}{1 - (a_0 R)^\beta} F. \quad (15)$$

We can identify the polarization 1-form $p = \frac{(a_0 R)^\beta}{1 - (a_0 R)^\beta} \iota_U F$ and the magnetization 1-form $m = -\frac{(a_0 R)^\beta}{1 - (a_0 R)^\beta} \iota_U * F$ related with the above theory where U is a unit, time-like velocity vector field associated with an inertial observer. One can find more information about these concepts in [14].

III. STATIC, SPHERICALLY SYMMETRIC SOLUTIONS

We look for static, spherically symmetric solutions to the field equations which are given by the metric

$$g = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \quad (16)$$

We consider a static electric potential 1-form $A = h(r)dt$. Then, electromagnetic field 2-form

$$F = dA = h' dr \wedge dt = H dr \wedge dt \quad (17)$$

has a spherically symmetric electric field component only.

We note that, for the minimal case $a_0 = 0$, the Reissner-Nordström metric

$$g = -\left(1 - \frac{2M}{r} + \frac{\kappa^2 q^2}{2r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{\kappa^2 q^2}{2r^2}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \quad (18)$$

and the Coulomb electric potential

$$A = -\frac{q}{r}dt \quad (19)$$

is a solution of (11) and (12).

After a lengthy calculation, the non-minimally coupled ($a_0 \neq 0$) Einstein-Maxwell equations (11) and (12) are reduced for the metric (16) and the electromagnetic 2-form (17). We do not find it useful to write down the reduced field equations in full detail, because they are very long and complicated. In order to avoid the problems of having higher order derivatives in our field equations, we restrict attention to those cases for which the Lagrange multipliers $\lambda_a = 0$. We calculate

$$\lambda_0 = -\lambda_2 = \lambda_3 = f \frac{d}{dr} \left(H^2 \frac{\beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} \right), \quad \text{and} \quad \lambda^1 = 0 \quad (20)$$

where the curvature scalar is given by

$$-R = f^{2''} + \frac{4f^{2'}}{r} + \frac{2}{r^2}(f^2 - 1). \quad (21)$$

Thus, we set

$$H^2 \frac{\beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} = \text{constant}. \quad (22)$$

The Maxwell equations (12) can be immediately integrated to give

$$\frac{1}{1 - (a_0 R)^\beta} H r^2 = q \quad (23)$$

where q is the electric charge determined by the Gauss integral

$$\frac{1}{4\pi} \int_{S^2} *G = q.$$

The expressions (22) and (23) imply the following relation for consistency:

$$R = a_1 r^{\frac{4}{\beta-1}} \quad (24)$$

where a_1 is an integration constant to be fixed. Then, it can be shown easily that the remaining Einstein field equations become

$$\frac{1}{2} \left(f^{2''} - \frac{2}{r^2}(f^2 - 1) \right) \left(\frac{1}{\kappa^2} + H^2 \frac{\beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} \right) - \frac{qH}{r^2} = 0, \quad (25)$$

$$\frac{R}{2} \left(1 - \frac{\kappa^2 \beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} H^2 \right) = 0. \quad (26)$$

The coupled equations (24)-(26) are solved by

$$\begin{aligned} f^2(r) &= 1 - \frac{C}{r} + \frac{\kappa^2 q^2}{4r^2} - \frac{a_1(\beta-1)^2}{4\beta(3\beta+1)} r^{\frac{2\beta+2}{\beta-1}} \quad \text{for } a_0 \neq 0, \quad \beta \neq 0, 1, -\frac{1}{3} \\ h(r) &= -\frac{q}{r} - \frac{a_1(\beta-1)}{\kappa^2 q \beta(3\beta+1)} r^{\frac{3\beta+1}{\beta-1}}. \end{aligned} \quad (27)$$

with $a_1 = (\kappa^2 q^2 \beta a_0^\beta)^{\frac{1}{1-\beta}}$. If we go back to non-minimally coupled Einstein-Maxwell field equations and check the solutions, we fix the integration constant in the (22) to be $\frac{1}{\kappa^2}$. We further note that the values of $\beta \neq 0$ in the range $-\frac{1}{3} < \beta < 1$ give asymptotically flat solutions. For β in this interval, we identify the mass $M = \frac{C}{2}$. Moreover, $r = 0$ is an essential singularity in general where the quadratic curvature invariant $*(R_{ab} \wedge *R^{ab}) \rightarrow \infty$ as $r \rightarrow 0$. It is remarkable to observe that the solution (27) goes to the Schwarzschild solution for $q = 0$, while for $q \neq 0$ but $a_0 = 0$ the metric in (27) does not coincide with the Reissner-Nordström metric (18). The explanation is as follows: For $a_0 = 0$, we have $R = 0$, so that (26) is identically satisfied. In this case, (25) is solved by the Reissner-Nordström metric. On the other hand, for $a_0 \neq 0$ and $a_1 \neq 0$ we solve (26) with the choice

$$\frac{\kappa^2 \beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} H^2 = 1$$

for which the limit $a_0 \rightarrow 0$ does not exist. We would like to point out that with the choice

$$\frac{\kappa^2 \beta a_0^\beta R^{\beta-1}}{(1 - (a_0 R)^\beta)^2} H^2 = 0$$

for an arbitrary $a_0 \neq 0$, we may set $a_1 = 0$ in (24) and show that our non-minimally coupled model admits the Reissner-Nordström black hole (18) and (19) as a solution.

A metric function similar to ours above with $\beta = \frac{1}{3}$ is obtained and analyzed in a recent paper [15]. They consider a different theory of non-minimally coupled electromagnetic fields to gravity. Furthermore, their electromagnetic field is that of a Dirac magnetic monopole Q_m . Contrary to our case, this solution joins smoothly to the Reissner-Nordström solution in the absence of non-minimal couplings. We also wish to point out that the metric (27) with $\beta = -3$ involves a Rindler acceleration term. This is noted by [16, 17] in a dilaton-gravity model and may be related with the source of various anomalies such as the rotation curves of spiral galaxies and the Pioneer anomaly.

IV. CONCLUSION

We have formulated non-minimal $R^\beta F^2$ -type coupled Einstein-Maxwell theory. The field equations are obtained by a first order variational principle using the method of Lagrange multipliers in the language of exterior differential forms. We found static, spherically symmetric solutions for arbitrary values of the β parameter. Such solutions may be related with the source of various anomalies in gravity for certain values of the parameters a_0 and β . Moreover, they may contribute to resolution of many important challenges such as dark matter, dark energy and Pioneer anomaly without any other fields. In addition, the non-minimal terms modify electromagnetic potentials at cosmological scales. Then the conventional electromagnetic energy density of the universe gets modified. In other words; if the effects of dark matter are not due to some exotic matter fields, the non-minimal couplings can lead to such effects[18]. Even if the electric charge q is not large, one may choose the parameters in such a way that these solutions can explain the rotation curves of galaxies and Pioneer anomaly for some parameter values.

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